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Aequationes Mathematicae

A refinement of the right-hand side of the Hermite–Hadamard inequality for simplices

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Abstract. We establish a new refinement of the right-hand side of the Hermite–Hadamard inequality for simplices, based on the average values of a convex function over the faces of a simplex and over the values at their barycenters.

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1. Introduction

The classical Hermite–Hadamard inequality [2] states that for a convex function $f\colon [a,b] \longrightarrow \mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

There are many generalizations. One of them ([1,5,6]) says that if $\Delta \subset \mathbb{R}^n$ is a simplex with barycenter $\mathbf{b}$ and vertices $\mathbf{x}_0,\ldots,\mathbf{x}_n$ and $f\colon \Delta \to \mathbb{R}$ is convex, then

$$f(\mathbf{b}) \leq \frac{1}{\mathrm{Vol}(\Delta)}\int_\Delta f(\mathbf{x})\,\mathrm{d}\mathbf{x} \leq \frac{f(\mathbf{x}_0)+\cdots+f(\mathbf{x}_n)}{n+1}. \tag{2}$$

In this paper we aim to improve the right-hand side of the last inequality.

Our main result can be formulated as follows: suppose we split the vertices of a simplex Δ into disjoint sets $K_i,\ i=1,\ldots,p$ and denote by Δ_{K_i} the face

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of Δ with vertices in K_i. Then for every convex function f defined on Δ we have

$$\operatorname{Avg}(f,\Delta) \le \sum_{i=1}^{p} \frac{\dim \Delta_{K_i}+1}{\dim \Delta + 1} \operatorname{Avg}(f,\Delta_{K_i}), \tag{3}$$

where $\operatorname{Avg}(f,\Delta)$ denotes the average value of f on the simplex with respect to an appropriate Lebesgue measure. We analyze the relationship between different right-hand sides of (3).

We also provide a class of refinements involving the values of the function f at barycenters of faces.

2. Definitions and lemmas

For a fixed natural number $n \ge 1$ let $N = \{0,1,\dots,n\}$. Suppose $\mathbf{x}_0,\dots,\mathbf{x}_n \in \mathbb{R}^n$ are such that the vectors $\overrightarrow{\mathbf{x}_0\mathbf{x}_i}$, $i=1,\dots,n$ are linearly independent.

Suppose K is a nonempty subset of N with $k+1$ elements $(0 \le k \le n)$ and denote the elements of $\{\mathbf{x}_i\colon i \in K\}$ by $\{\mathbf{y}_0,\dots,\mathbf{y}_k\}$. The set $\Delta_K = \operatorname{conv}\{\mathbf{y}_0,\dots,\mathbf{y}_k\}$ is called a *simplex* (or a *k-simplex* if we want to emphasize its dimension). The n-simplex Δ_N will be denoted by Δ.

Given two sets $K \subset L \subset N$ the simplex Δ_K is called a *face* (or *k-face*) of Δ_L.

The points in Δ_K admit a unique representation of the form

$$\mathbf{y} = \sum_{i=0}^{k} \alpha_i \mathbf{y}_i, \quad \alpha_i \ge 0, \quad \sum_{i=0}^{k} \alpha_i = 1.$$

The $k+1$-tuple $(\alpha_0,\dots,\alpha_k)$ is called *barycentric coordinates*. The point

$$\mathbf{b}_K = \frac{1}{k+1}(\mathbf{y}_0 + \cdots + \mathbf{y}_k)$$

is called the *barycenter* of Δ_K.

If Σ is a k-dimensional simplex and $f : \Sigma \to \mathbb{R}$ is an integrable function, then we shall denote its average value over the simplex by $\operatorname{Avg}(f,\Sigma)$, i.e.

$$\operatorname{Avg}(f,\Sigma) = \frac{1}{\operatorname{Vol}(\Sigma)} \int_{\Sigma} f(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

The integration here is with respect to the k-dimensional Lebesgue measure denoted by $\mathrm{d}\mathbf{x}$ and $\operatorname{Vol}(\Sigma)$ denotes the k-dimensional volume.

With this notation we can write the right-hand side of (2) as

$$\operatorname{Avg}(f,\Delta) \le \frac{1}{n+1} \sum_{i=0}^{n} \operatorname{Avg}(f,\Delta_{\{i\}}). \tag{4}$$

The set $E_k = \{(\alpha_1,\dots,\alpha_k)\colon \alpha_i \ge 0, \sum_{i=1}^{k} \alpha_i \le 1\} \subset \mathbb{R}^k$ is called a *standard simplex*.

For every k-simplex Δ_K we define a one-to-one mapping $\varphi_K\colon E_k \to \Delta_K$ given by the formula

$$\varphi_K(\alpha_1, \ldots, \alpha_k) = \mathbf{y}_0 + \sum_{i=1}^{k} \alpha_i(\mathbf{y}_i - \mathbf{y}_0).$$

Note that $\partial\varphi_K/\partial\alpha_i = \mathbf{y}_i - \mathbf{y}_0$, so the absolute value of its Jacobian equals $|\det D\varphi_K| = k!\mathrm{Vol}(\Delta_K)$. This means, that if $g\colon \Delta_K \to \mathbb{R}$ is an integrable function, then the identity

$$\frac{1}{\mathrm{Vol}(\Delta_K)} \int_{\Delta_K} g(\mathbf{x})\,\mathrm{d}\mathbf{x} = k! \int_{E_k} g(\varphi_K(\boldsymbol{\alpha}))\,\mathrm{d}\boldsymbol{\alpha} \tag{5}$$

holds. In particular $\mathrm{Vol}(E_k) = 1/k!$.

A *partition* of N is a set

$$\mathcal{K} = \{K_i : i = 1 \ldots p,\ K_i \subset N,\ \bigcup_{i=1}^{p} K_i = N,\ K_i \cap K_j = \emptyset\}.$$

We say that $\mathcal{L}$ *refines* $\mathcal{K}$ (and write $\mathcal{L} \prec \mathcal{K}$) if every element of $\mathcal{L}$ is a subset of an element of $\mathcal{K}$.

3. Faces based refinements

The inequality (4) shows that the average of a convex function over a simplex can be bounded by the convex combination of its average values over its 0-faces. Our main result—Theorem 3.1 and its corollaries generalize this fact.

Theorem 3.1. *Let $\mathcal{K} = \{K_1, \ldots, K_p\}$ be a partition of N and $f\colon \Delta \to \mathbb{R}$ be a convex function. Define*

$$F(\mathcal{K}) = \sum_{i=1}^{p} \mathrm{card}\, K_i\, \mathrm{Avg}(f, \Delta_{K_i}).$$

If $\mathcal{L} \prec \mathcal{K}$, then

$$F(\mathcal{K}) \le F(\mathcal{L}). \tag{6}$$

To prove this theorem we shall need Lemma 3.2.

Lemma 3.2. *Let $K, L \subset N$ be two disjoint, nonempty sets with* $\mathrm{card}\, K = k + 1$ *and* $\mathrm{card}\, L = l + 1$. *Further, let $f\colon \Delta \to \mathbb{R}$ be a convex function. Then*

$$\mathrm{Avg}(f, \Delta_{K\cup L}) \le \frac{k+1}{k+l+2} \cdot \mathrm{Avg}(f, \Delta_K) + \frac{l+1}{k+l+2} \cdot \mathrm{Avg}(f, \Delta_L).$$

Proof. Denote by $\mathbf{u}_0, \ldots, \mathbf{u}_k$ the vertices of Δ_K and by $(\alpha_0, \ldots, \alpha_k)$ its barycentric coordinates. Similarly, let $\mathbf{v}_0, \ldots, \mathbf{v}_l$ be the vertices of Δ_L and $(\beta_0, \ldots, \beta_l)$ be its barycentric coordinates. Every point $\mathbf{x} \in \Delta_{K\cup L}$ can be represented as

$$\mathbf{x} = \sum_{i=0}^{k} \alpha_i' \mathbf{u}_i + \sum_{j=0}^{l} \beta_j' \mathbf{v}_j,\ \alpha_i', \beta_j' \geq 0,\ \sum_{i=0}^{k} \alpha_i' + \sum_{j=0}^{l} \beta_j' = 1.$$

Let $\sum_{i=0}^{k} \alpha_i' = s,\ \sum_{j=0}^{l} \beta_j' = 1 - s$ and $\alpha_i = \alpha_i'/s,\ \beta_j = \beta_j'/(1-s)$ (if division by zero occurs we assume the result is zero). Then

$$\mathbf{x} = s \sum_{i=0}^{k} \alpha_i \mathbf{u}_i + (1-s) \sum_{j=0}^{l} \beta_j \mathbf{v}_j.$$

Let $\Phi\colon [0,1] \times E_k \times E_l \to \Delta_{K\cup L}$ be defined by

$$\begin{aligned}
&\Phi(s,\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l) \\
&\quad = s\left(\mathbf{u}_0 + \sum_{i=1}^{k} \alpha_i(\mathbf{u}_i - \mathbf{u}_0)\right) + (1-s)\left(\mathbf{v}_0 + \sum_{j=1}^{l} \beta_j(\mathbf{v}_j - \mathbf{v}_0)\right) \\
&\quad = s\varphi_K(\alpha_1, \ldots, \alpha_k) + (1-s)\varphi_L(\beta_1, \ldots, \beta_l).
\end{aligned}$$

Its Jacobian equals

$$\begin{aligned}
\det D\Phi &= \begin{vmatrix} \Phi_s' \\ \vdots \\ \Phi_{\alpha_i}' \\ \vdots \\ \Phi_{\beta_j}' \\ \vdots \end{vmatrix} = \begin{vmatrix} \mathbf{u}_0 - \mathbf{v}_0 + \sum_{i=1}^{k} \alpha_i(\mathbf{u}_i - \mathbf{u}_0) - \sum_{j=1}^{l} \beta_j(\mathbf{v}_j - \mathbf{v}_0) \\ \vdots \\ s(\mathbf{u}_i - \mathbf{u}_0) \\ \vdots \\ (1-s)(\mathbf{v}_j - \mathbf{v}_0) \\ \vdots \end{vmatrix} \\
&= s^k(1-s)^l \begin{vmatrix} \mathbf{u}_0 - \mathbf{v}_0 \\ \vdots \\ \mathbf{u}_i - \mathbf{u}_0 \\ \vdots \\ \mathbf{v}_j - \mathbf{v}_0 \\ \vdots \end{vmatrix} = s^k(1-s)^l \begin{vmatrix} \mathbf{u}_0 - \mathbf{v}_0 \\ \vdots \\ \mathbf{u}_i - \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_j - \mathbf{v}_0 \\ \vdots \end{vmatrix} \\
&= \pm(k+l+1)! s^k (1-s)^l \mathrm{Vol}(\Delta_{K\cup L}).
\end{aligned} \tag{7}$$

Changing the variable in the integral yields

$$\int_{\Delta_{K\cup L}} f(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_0^1 \mathrm{d}s \int_{E_k} \mathrm{d}\boldsymbol{\alpha} \int_{E_l} \mathrm{d}\boldsymbol{\beta} f(\Phi(s, \boldsymbol{\alpha}, \boldsymbol{\beta}))|\det D\Phi|. \tag{8}$$

Using the convexity of f and the formula (5) we obtain

$$\begin{aligned}
&\int_0^1 \mathrm{d}s \int_{E_k} \mathrm{d}\boldsymbol{\alpha} \int_{E_l} \mathrm{d}\boldsymbol{\beta} f(\Phi(s,\boldsymbol{\alpha},\boldsymbol{\beta})) s^k (1-s)^l \\
&\qquad = \int_0^1 s^k (1-s)^l \,\mathrm{d}s \int_{E_k} \mathrm{d}\boldsymbol{\alpha} \int_{E_l} \mathrm{d}\boldsymbol{\beta} f(s\varphi_K(\boldsymbol{\alpha}) + (1-s)\varphi_L(\boldsymbol{\beta})) \\
&\qquad \leq \quad \int_0^1 s^{k+1}(1-s)^l \,\mathrm{d}s \int_{E_k} f(\varphi_K(\boldsymbol{\alpha}))\,\mathrm{d}\boldsymbol{\alpha} \int_{E_l} \mathrm{d}\boldsymbol{\beta} \\
&\qquad\quad + \int_0^1 s^k (1-s)^{l+1} \,\mathrm{d}s \int_{E_k} \mathrm{d}\boldsymbol{\alpha} \int_{E_l} f(\varphi_L(\boldsymbol{\beta}))\,\mathrm{d}\boldsymbol{\beta} \\
&\qquad = \frac{(k+1)!l!}{(k+l+2)!} \cdot \frac{\operatorname{Avg}(f,\Delta_K)}{k!} \cdot \frac{1}{l!} + \frac{k!(l+1)!}{(k+l+2)!} \cdot \frac{1}{k!} \cdot \frac{\operatorname{Avg}(f,\Delta_L)}{l!}. \qquad (9)
\end{aligned}$$

The Lemma follows from (7), (8) and (9). □

Now we can prove our main result.

Proof of Theorem 3.1. It follows from Lemma 3.2 that the function

$$K \mapsto \operatorname{card} K \operatorname{Avg}(f,\Delta_K)$$

is subadditive on disjoint sets, so (6) follows by mathematical induction on the cardinality of partition. □

Since $\{\{0\},\dots,\{n\}\} \prec \mathcal{K} \prec \{N\}$ for any partition $\mathcal{K}$, we obtain the refinements of the Hermite–Hadamard inequality.

Corollary 3.3. *For any partition $\mathcal{K}$ the inequalities*

$$(n+1)\operatorname{Avg}(f,\Delta) \leq F(\mathcal{K}) \leq f(\mathbf{x}_0) + \cdots + f(\mathbf{x}_n)$$

hold.

Applying Corollary 3.3 to all possible pairs consisting of a vertex and its opposite face one obtains the following result.

Corollary 3.4. *Let $f\colon \Delta \to \mathbb{R}$ be a convex function. Then the inequality*

$$\begin{aligned}
\operatorname{Avg}(f,\Delta) \leq{}& \frac{1}{n+1}\frac{f(\mathbf{x}_0) + \cdots + f(\mathbf{x}_n)}{n+1} \\
&+ \frac{n}{n+1} \cdot \frac{1}{n+1} \sum_{\substack{K\subset N \\ \operatorname{card} K = n}} \operatorname{Avg}(f,\Delta_K)
\end{aligned}$$

holds.

If N can be divided into disjoint subsets of the same cardinality, then applying Corollary 3.3 to all possible partitions and summing the obtained inequalities one gets the following corollary.

Corollary 3.5. *Let $f\colon \Delta \to \mathbb{R}$ be a convex function and d be a divisor of $n+1 = \operatorname{card} N$. Then*

$$\operatorname{Avg}(f,\Delta) \le \frac{1}{\binom{n+1}{d}} \sum_{\substack{K\subset N\\ \operatorname{card} K=d}} \operatorname{Avg}(f,\Delta_K).$$

It is known (see e.g. [7, p. 125]) that the volume of a regular simplex $\Delta \subset \mathbb{R}^n$ with unit edges equals $\frac{1}{n!}\sqrt{\frac{n+1}{2^n}}$. In this case Corollary 3.5 yields

Corollary 3.6. *Let f be a convex function defined on a regular simplex Δ with unit edges. Let d be a divisor of $n+1 = \operatorname{card} N$. Then*

$$\int_\Delta f(\mathbf{x})\,\mathrm{d}\mathbf{x} \le \left[\binom{n}{d-1}\right]^{-2} \frac{1}{(n+1-d)!}\sqrt{\frac{d}{n+1}\frac{1}{2^{n+1-d}}} \sum_{\substack{K\subset N\\ \operatorname{card} K=d}} \int_{\Delta_K} f(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

4. Barycentric refinements

It is known that the right-hand side of the Hermite–Hadamard inequality can be refined as follows ([3,4,8]):

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right],$$

or in the case of a simplex

$$\operatorname{Avg}(f,\Delta) \le \frac{1}{n+1} f(\mathbf{b}) + \frac{n}{n+1}\,\frac{f(\mathbf{x}_0)+\cdots+f(\mathbf{x}_n)}{n+1}. \tag{10}$$

Given the fact that the vertices are in fact the barycenters of 0-faces, we see that in this case the average value of f over a simplex is bounded by a convex combination of its values at the barycenters of all faces.

The next theorem improves (10) and can be a base for further refinements.

Theorem 4.1. *Let $\mathbf{b}$ be the barycenter of Δ and let $K_i = N\setminus\{i\}$. If $f:\Delta\to\mathbb{R}$ is convex, then*

$$\operatorname{Avg}(f,\Delta) \le \frac{1}{n+1} f(\mathbf{b}) + \frac{n}{n+1}\cdot\frac{1}{n+1}\sum_{i=0}^{n} \operatorname{Avg}(f,\Delta_{K_i}).$$

Proof. Note that we can identify the partitions of N with partitions of the set of vertices of a simplex. The barycenter divides Δ into $n+1$ simplices

$$B_i = \operatorname{conv}\{\mathbf{x}_0,\ldots,\mathbf{x}_{i-1},\mathbf{b},\mathbf{x}_{i+1},\ldots,\mathbf{x}_n\},\quad i=0,\ldots,n.$$

For each of them we split its vertices into two groups: $\{\mathbf{b}\}$ and $\{\mathbf{x}_j : j \neq i\}$ and apply Lemma 3.2 to these partitions. Taking into account that $\operatorname{Vol}(B_i) = \frac{1}{n+1}\operatorname{Vol}(\Delta)$ for every i, we get

$$\begin{aligned}\frac{\int_{B_i} f(\mathbf{x})\,\mathrm{d}\mathbf{x}}{\operatorname{Vol}(\Delta)} &= \frac{1}{n+1}\operatorname{Avg}(f, B_i) \\ &\leq \frac{1}{n+1}\left[\frac{1}{n+1} f(\mathbf{b}) + \frac{n}{n+1}\operatorname{Avg}(f, \Delta_{K_i})\right]. \qquad (11)\end{aligned}$$

Summing these inequalities completes the proof. □

Now one can see that applying (2) to all simplices in (11) we obtain (10). But we can do much better: instead of (2) we can use Lemma 4.1 recursively to $(n-1)$-faces etc. and continue this process until we reach 0-faces. Thus we obtain

Theorem 4.2. *Under the assumptions of Theorem 4.1*

$$\operatorname{Avg}(f, \Delta) \leq \frac{1}{n+1}\sum_{k=1}^{n+1}\frac{1}{\binom{n+1}{k}}\sum_{\substack{K \subset N \\ \operatorname{card} K = k}} f(\mathbf{b}_K).$$

Combining the results of Corollary 3.3 and Theorem 4.1 one can produce various new upper bounds for the average value of f over the simplex. Below we show one of them.

Splitting the vertices of Δ into the maximum number of groups of k elements and one group of l elements, where $l < k$, and taking the average over all such splits one gets

Corollary 4.3. *Let $\Delta \subset \mathbb{R}^n$ be an arbitrary simplex and let $\mathbf{x}_0, \dots, \mathbf{x}_n$ be its vertices.*

Then for every convex function $f : \Delta \to \mathbb{R}$ we have

$$\operatorname{Avg}(f, \Delta) \leq \alpha_n \frac{1}{\binom{n+1}{k}}\sum_{\substack{K \subset N \\ \operatorname{card} K = k}} f(\mathbf{b}_K) + (1 - \alpha_n)\frac{f(\mathbf{x}_0) + \cdots + f(\mathbf{x}_n)}{n+1},$$

where $\alpha_n = \frac{\lfloor \frac{n+1}{k} \rfloor}{n+1}$.

In particular we obtain this corollary.

Corollary 4.4. *Let $\Delta \subset \mathbb{R}^n$ be an arbitrary simplex with vertices $\mathbf{x}_0, \dots, \mathbf{x}_n$. Then for every convex function $f : \Delta \to \mathbb{R}$ we have*

$$\operatorname{Avg}(f, \Delta) \leq \alpha_n \frac{1}{\binom{n+1}{2}}\sum_{0 \leq i < j \leq n} f\left(\frac{\mathbf{x}_i + \mathbf{x}_j}{2}\right) + \beta_n \frac{f(\mathbf{x}_0) + \cdots + f(\mathbf{x}_n)}{n+1},$$

where $\alpha_n = \frac{\lfloor \frac{n+1}{2} \rfloor}{n+1}$ and $\beta_n = \frac{\lceil \frac{n+1}{2} \rceil}{n+1}$.

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